Homework 2

This homework is worth 100 points in total. The answers **have to be typed, and submitted via Blackboard, BUT if you have difficulties in drawing models in LaTEX you man attach scans of drawings done by hand**. You can answer them *either individually*, or in *pairs* or *small groups* of (**at most 3-4 people**). **Only one homework per group should be submitted, but the names of everybody in the group should be written on top of the paper**. **The names of everybody in the group should be written on top of the paper**.

Exercise 1. (*32 points*) In a far away country, the Queen is giving distinctions of honor by placing hats on the heads of remarkable people. Alice and Bob are each due to receive a distinction. It is common knowledge that the Queen has only three hats left, namely 2 red hats and 1 white hat. The Queen asks Alice and Bob to close their eyes, and then places a hat on each of their heads. Let us suppose that in fact the queen chooses to place *red* hats on *both* their heads. Then they are allowed to look: each can see the other's hat, but not his/her own.

1. (*8 points*) Write a sentence θ in epistemic logic encoding *all* of the above information (including the real color of the children's hats, the knowledge of Alice and Bob about each other's hats and about themselves, the Queen's knowledge and, most importantly, everything that is the common knowledge!).

2. (*8 points*) Represent the above situation as a state model $\mathbf{M}$, for a set $\mathcal{A} = \{a, b, q\}$ of three agents (Alice, Bob and the Queen) and a set $\Phi = \{r_a, w_a, r_b, w_b\}$ of four atomic sentences (describing the colors of Alice's and Bob's hats).

   Is this an *epistemic* state model?

3. (*8 points*) Now the following action happens: secretly and separately from each other, Alice and Bob look in their mirrors and see their own (*red*!) hats; **none of them suspects that the other one is doing this**; and each of them *knows that the other doesn't suspect anything.*

Moreover, each of them *believes* that the queen doesn't suspect them either. But in fact, the queen can see what both of them are doing. (So she knows they're looking in the mirror. In fact, she knows everything that's going on!)

Represent all this scenario using a *event model* $\mathbf{\Sigma}$, with 4 actions.

Is this an *epistemic* (event) model? Is it a *doxastic* (event) model?

4. (*8 points*) Compute the *update product* $\mathbf{M} \otimes \mathbf{\Sigma}$ of the two models, and draw the resulting state model, representing the situation *after* the action described in the previous part.

   Is this an *epistemic* (event) model? Is it a *doxastic* (event) model?

Exercise 2. (*36 points*)

Consider the following scenario: Alice, Bob and Charles are given a "bit" (i.e. a number from the set $\{0, 1\}$. It is common knowledge that: (a) the sum of all their bits is even, (b) each of the agents knows only his/her own bit, but not the others' bits. Let us assume that in reality Alice is given number 1, Bob receives number 0, and Charles receives number 1 as well.

1. (4 points) Represent (draw) this situation as an epistemic model M, with three agents (a for Alice, b for Bob, c for Charles), denoting the possible worlds by ordered triples (x_a, x_b, x_c), with $x_i \in \{0, 1\}$(e.g. the world $(0, 1, 0)$ is one in which Alice received 0, Bob received 1 and Charles received 0). Draw the epistemic accessibility relations for each agent. Also, specify the valuation for a language having atomic sentences of the form $x_i = \beta$, with $\beta \in \{0, 1\}$, and $i \in \{a, b, c\}$ (where e.g. $x_a = 0$ means that Alice received number 0), and specify which possible world is the "real" one.

2. (*8 points*) Suppose now the following action happens: *Bob takes a peek at Alice's bit and sees that it is a 1* (i.e. $x_a = 1$). He *thinks that nobody noticed this*: indeed, he believes that the others do not suspect him of peeking. He is right about Charles: indeed, *Charles doesn't suspect anything.* However, Alice **does** notice that Bob was peeking at her bit! Moreover, Alice realizes that Bob thinks he fooled everybody: so she *knows* that Bob believes nobody noticed his secret peeking. Finally, she also knows that Charles really didn't suspect any of this going on.

   Represent (draw) this action using an *event model* $\mathbf{\Sigma}$ with 3 actions. (Specify the actions' preconditions, draw relations representing agents' beliefs about what is going on, and specify which action is the "real" one.) *Is this an epistemic model, a doxastic model or none of the two*?

3. (*8 points*) *Represent (draw) a model* $\mathbf{M}'$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}' = \mathbf{M} \otimes \mathbf{\Sigma}$. Specify which world is *the "real" one.*

4. (*8 points*) Let us now consider an *alternative scenario*: as in the first scenario, *Bob takes a peek at Alice's number and sees that it is* 1; but this time, **both Alice and Charles notice** (that he is peeking at Alice's bit). Moreover, Bob's peaking is so obvious, that Alice knows that Charles noticed, and Charles knows that Alice noticed etc: indeed, Bob's peeking (i.e. the fact that Bob was taking a peek at Alice's number) is **common knowledge between Alice and Charles**. However, Bob is completely unaware of all this; as before, he *thinks that nobody noticed his peeking.* And in fact, this (Bob's unawareness) is also common knowledge between Alice and Charles: they know, and know that the other of them knows etc, that Bob thinks he fooled them.

   Draw an event model $\mathbf{\Sigma}'$ *representing this action.* (As before, specify the preconditions, draw the relations R_1, R_2, R_3 and specify which action is the "real" one.)

   HINT: The event model has 5 actions. Note that Charles knows that Bob took a peek and saw Alice's number, but he doesn't know *what* Bob saw: Charles cannot see Alice's number!

5. (*8 points*) Let us assume now that, starting *in the initial situation* (described by the model $\mathbf{M}$ in part 1 above), the scenario described in part 4 (the obvious peeking by Bob, noticed by both Alice and Charles) is happening now. Represent (draw) a model of the situation after this action, by computing the update product $\mathbf{M}'' = \mathbf{M} \otimes \mathbf{\Sigma}'$ (of the *initial* model $\mathbf{M}$ with the event model from part 4).

Exercise 3 (*32 points*) Each of two children, Alice and Bob, has a natural number written in the *back of his/her head.* Let $n_a \in \{0, 1, 2, \ldots\}$ be Alice's number and $n_b \in \{0, 1, 2, \ldots\}$ be Bob's number It is common knowledge that: (i) *no child can see his/her own number*, (ii) Alice stands in the back of Bob, so *she can see Bob's number* n_b*, but Bob cannot see any of the numbers*, and (iii) one of the numbers is the immediate successor of the other (in any order): i.e. either $n_a = n_b + 1$ or $n_b = n_a + 1$.

In the following, each of the children will be asked a number of questions, which they are required to answer publicly and truthfully.

1. (*2 points*) *How many possible worlds* are there (that are consistent with the story above)?

2. (*3 points*) *Represent* **(draw) the above situation as an *epistemic model*** $\mathbf{M}_1$, with two agents (a for Alice, b for Bob), using pairs of numbers (n_a, n_b) as "names" for the possible worlds. **Draw the epistemic accessibility relations** for each agent. but do *not* worry about the valuation (yet), since no atomic sentences are given yet.

3. (*3 points*) For your model, consider now the following *four atomic sentences* $0_a, 0_b, 1_a, 1_b$. Here, 0_a means "Alice's number is equal to 0 " (i.e. $n_a = 0$), and 0_b means "Bob's number is equal to 0 " (i.e. $n_b = 0$); 1_a means "Alice's number is equal to 1 " (i.e. $n_a = 1$), and 1_b means "Bob's number is equal to 1 " (i.e. $n_b = 1$). **Specify the valuation for these atomic sentences** in the above model.

4. (*4 points*) Alice is asked the following question: "*Do you know whether your own number is equal to* 0 *or not, and if so then which of the two*?" So her answers can be: (a) *I don't know*, (b) *I know that my number is equal to* 0, or (c) *I know that my number is NOT equal to* 0.

   Let us suppose that in fact *Alice answers (c) "I know that my number is NOT equal to* 0*"*.

   Write down a sentence ϕ in epistemic logic that expresses her *answer*.

5. (*4 points*) Interpreting the above answer as a truthful public announcement $!\phi$ of the sentence written in the previous part, **represent (draw) the updated model $\mathbf{M}_2 = \mathbf{M}_1{}^{!\phi}$ after this public announcement**.

6. (*4 points*) Suppose now that, *after* Alice answered as above, Bob is asked the "same" question: "*Do you (Bob) know whether your own number is equal to* 0 *or not, and if so then which of the two*?"

   What will Bob answer? Also, **what will be updated model $\mathbf{M}_3$ representing the situation after he answers**? Justify your answers!

7. (*4 points*) After the previous round of questions, Alice is now asked the following question: "*Do you know whether your own number is equal to* 1 *or not, and if so then which of the two*?" The answers can be: (a) *I don't know*, (b) *I know that my number is equal to* 1, or (c) *I know that my number is NOT equal to* 1.

   Let us suppose that in fact *Alice answers (a) "I don't know"*.

   Write down a sentence ψ in epistemic logic that expresses her *answer*.

8. (*4 points*) Interpreting the above answer as a truthful public announcement $!\psi$ of the sentence written in the previous part, **represent (draw) the updated model $\mathbf{M}_4 = \mathbf{M}_3{}^{!\psi}$ after this public announcement**.

9. (*4 points*) Suppose now that, *after* Alice answered as above, Bob is asked the "same" question: "*Do you (Bob) know whether your own number is equal to* 1 *or not, and if so then which of the two*?"

   What will Bob answer? Also, **what is his number**? Justify your answers!