ICS - ODE

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1. **Separable**: These are ODEs where the two variables can be separated such that each only appears on one side of the equals sign. Example:

$$\frac{dx}{dt} = x$$
$$\frac{1}{x} = 1dt$$

   This allows for solving this ODE using integration.
   Linear:
   First/second order:

2. **Explicit/implicit solutions**:
   Fixed points: These are points where the value of $x(t)$ (and other parameters) causes $\frac{dx}{dt}$ to be 0. This means that this causes $x(t)$ to remain constant.
   Locally/globally stable and unstable fixed points: With locally stable fixed points, the fixed point is convergent from an already close proximity. A globally fixed point is convergent regardless of whether there is a close proximity. UNstable fixed points are always divergent (unless you're on this fixed point of course).

3. The following analytical solutions are derived:

$$\frac{dx}{dt} = 1 \implies x(t) = t$$

$$\frac{dx}{dt} = 2t \implies x(t) = t^2 - 4$$
$$\frac{dx}{dt} = -x \implies x(t) = 4e^{-t}$$

   As can be seen in figure 1, ...

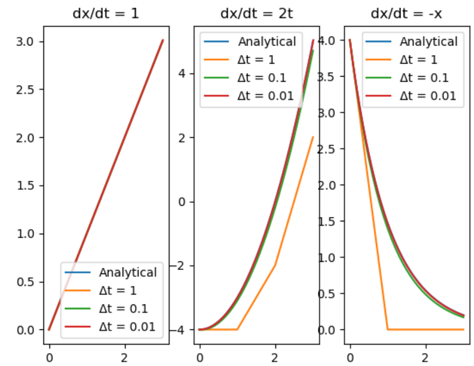

Figure 1:

4. (a) g is the coefficient denoting the constant generation of proteins and k denotes the constant degradation of proteins.

   (b)

$$\frac{dx}{dt} = g - kx$$
$$\frac{1}{g-kx}dx = 1dt$$
$$\int \frac{1}{g-kx}dx = \int 1dt$$
$$\frac{1}{-k}\ln|g-kx| = t + C$$
$$\pm(g-kx) = e^{-kt+-kC}$$
$$g - kx = Ce^{-kt}$$
$$-kx = Ce^{-kt} - g$$
$$x = Ce^{-kt} + \frac{g}{k}$$

   therefore:

$$x(t) = Ce^{-kt} + \frac{g}{k}$$

   (c) Additional constant C depends on the concentration at $t = 0$.

(d)

$$
\begin{aligned}
x(0) = C \cdot e^{-k \cdot 0} + \frac{g}{k} &= 0 \\
C \cdot e^{0} + \frac{g}{k} &= 0 \\
C + \frac{g}{k} &= 0 \\
C &= -\frac{g}{k}
\end{aligned}
$$

therefore:

$$x(t) = \frac{-g \cdot e^{-kt} + g}{k}$$

(e)

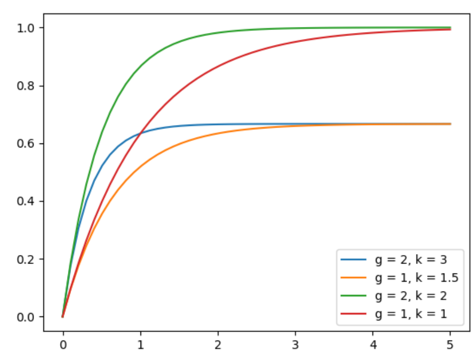

Figure 2:

(f)

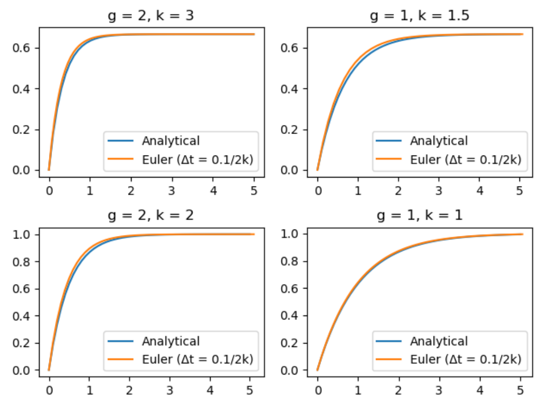

Figure 3:

(g)

$$\frac{dx}{dt} = g - kx = 0$$
$$kx = g$$
$$x = \frac{g}{k}$$

So in general, $x(t)$ settles to $x(t) = \frac{g}{k}$

(h) Figure 4 shows that for all tested values, $x(t)$ eventually settles on $\frac{g}{k}$. Therefore $x(t)$ is globally stable.

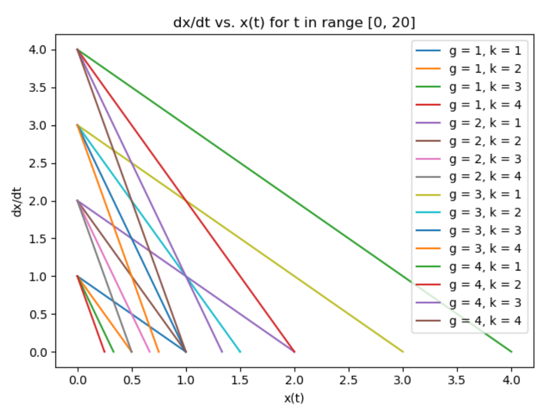

Figure 4:

(i) At first, g increases rapidly to compensate for the low concentration. After the deficit has been restored g decreases slightly to a value which is able to maintain equilibrium.

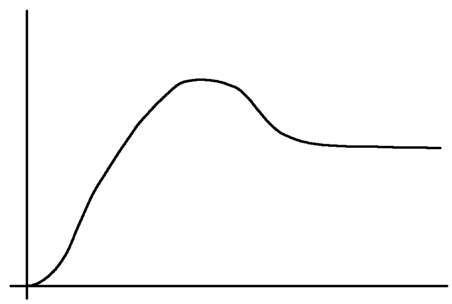

Figure 5:

(j) This is called mean field approximation because the model does not look at individual rabbits but at a field of rabbits as a whole.

5. (a) r is multiplied by x, causing an increase of rabbits equal to the population times reproduction rate per time step and k is multiplied by x and to the right of a subtraction, causing a decrease of rabbits equal to the population times dying rate per time step. These together form the net population change per time step.

(b)

$$\frac{dx}{dt} = rx - kx$$
$$\frac{1}{rx - kx} dx = 1dt$$
$$\int \frac{1}{rx - kx} dx = \int 1dt$$
$$\frac{1}{r - k} \ln |rx - kx| = t + C$$
$$\ln |rx - kx| = t(r - k) + C$$
$$\pm(rx - kx) = Ce^{t(r-k)}$$
$$x = Ce^{t(r-k)}$$

therefore:

$$x(t) = Ce^{t(r-k)}$$

(c) Figure 6 shows the three typical behaviours of $x(t)$. The fixed points of $x(t)$ is $x(t) = Cifr = k$. $x(t)$ (globally stable) and $x(t) = 0$ (globally stable if $r < k$, unstable if $r \geq k$).

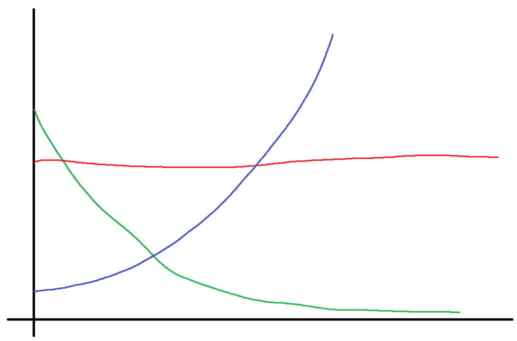

Figure 6: Red: $r = k$, Green: $r < k$ and Blue: $r > k$

(d)

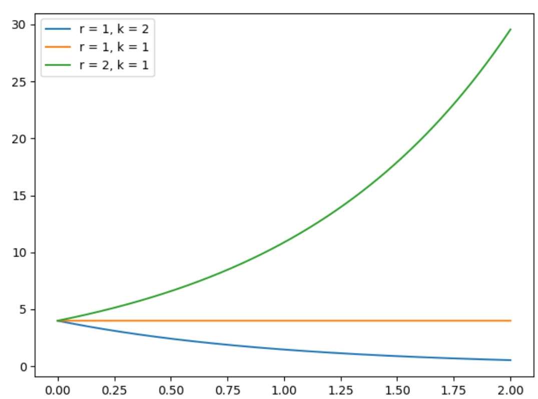

Figure 7: Euler approximation of x(t) for t in range [0, 2] with $x_0 = 4$

(e) Both $x(t) = C$ if $r = k$ and $x(t) = 0$.

(f) Due to the multiplication of r with x^2, the population now grows polynomially. This means that if x increases, the term with which r is multiplied with an increasingly larger term x^2

(g) —

(h)

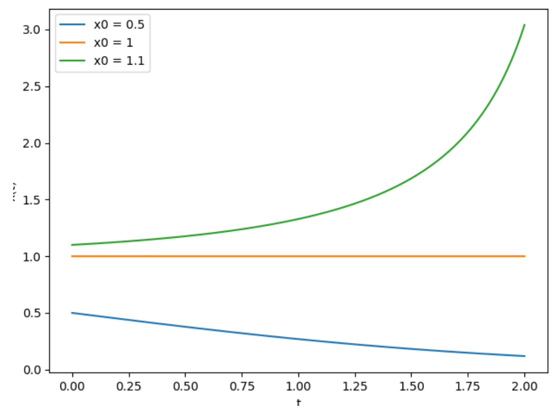

Figure 8:

(i) In this scenario, x_{max} is the maximum sustainable population size.

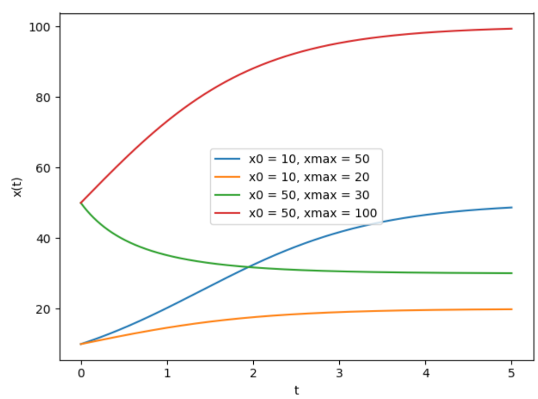

Figure 9:

(j) This model implements population dynamics with the factor $1 - \frac{x}{x_{max}}$. This factor gets closer to 0 as the population closes in on the maximum population size; meaning that the increase (or decrease if the population started by exceeding the maximum population size) in population decreases.

(k)

$$\frac{dx}{dt} = x(1 - \frac{x}{x_{max}}) = 0$$

therefore:

$$x = 0$$

$$1 - \frac{x}{x_{max}} = 0$$
$$\frac{x}{x_{max}} = 1$$
$$x = x_{max}$$

Thus, the fixed point are $x = 0$ (unstable) and $x = x_{max}$ (globally stable).

(l) Figure 10 shows that $\frac{dx}{dt} = 0$ for $x = 0$ and $x = 30$ (i.e. x_{max}). For points below fixed point $x = 0$ $\frac{dx}{dt}$ is negative and above this fixed point $\frac{dx}{dt}$ is positive. This proves that $x+0$ is an unstable fixed point. In the case of $x = x_{max}$, below this point $\frac{dx}{dt}$ is positive and above this point $\frac{dx}{dt}$ is negative, proving that this fixed point is globally positive.

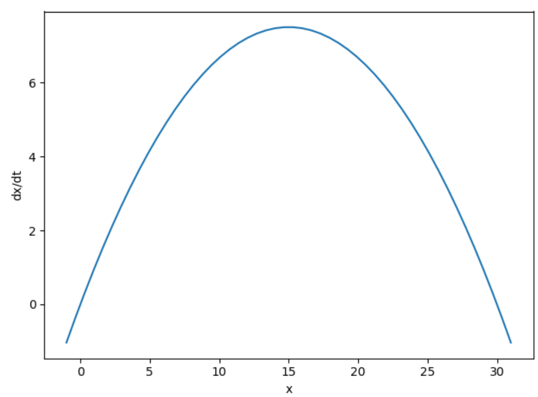

Figure 10: $\frac{dx}{dt}$ vs. x with $x_{max} = 30$

(m) The new ODE would look like:

$$\frac{dx}{dt} = x(1 - \frac{x}{x_{max}}) - rx$$

(n) This is true when:

$$\frac{dx}{dt} < 0$$
$$x(1 - \frac{x}{x_{max}}) - rx < 0$$
$$x(1 - \frac{x}{x_{max}}) < rx$$
$$(1 - \frac{x}{x_{max}}) < r$$

Yes, this makes sense.

(o) Due to a too coarse step size (see figure 11), the plot overshoots the x-axis and dips into the negative.

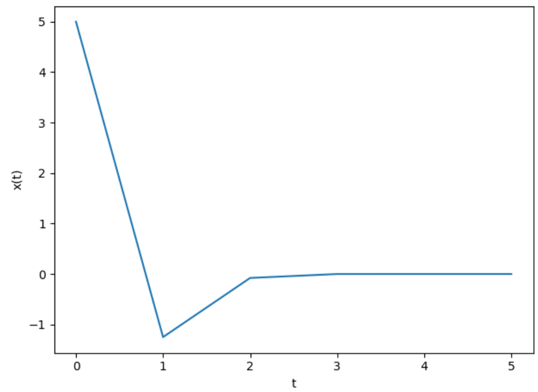

Figure 11: $\Delta t = 1$, $x_0 = 5$ and $x_{max} = 20$