Classical Cryptography

Basics: monoalphabetic substitution

Karst Koymans

Informatics Institute

University of Amsterdam

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1. The classic Caesar substitution cipher
   - Caesar's system
   - Alphabet encoding
   - Modular arithmetic
   - Mathematical formulation
   - Caesar cryptanalysis

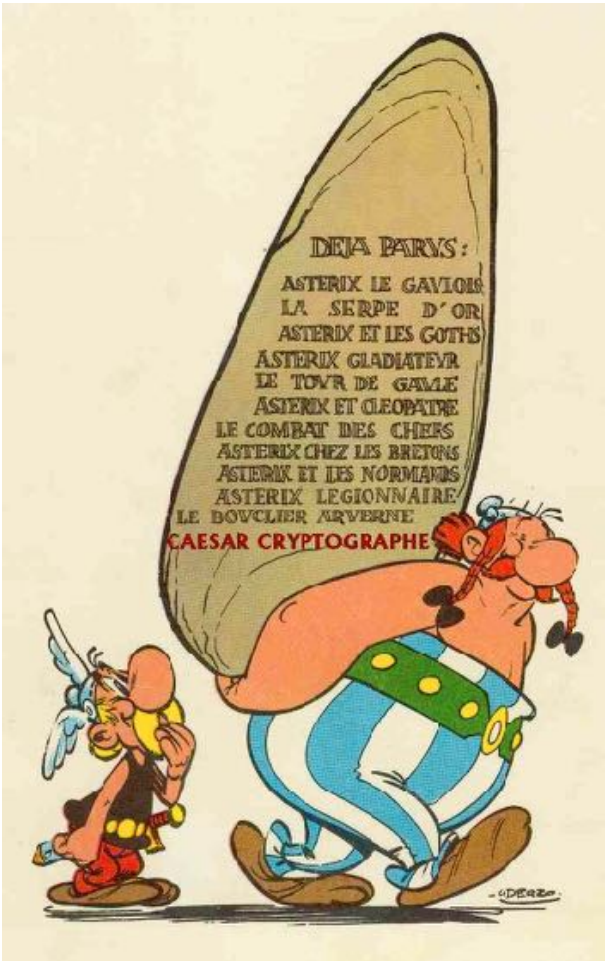
2. General monoalphabetic systems
   - Generating alphabets
   - Some number theory
   - Composition of ciphers

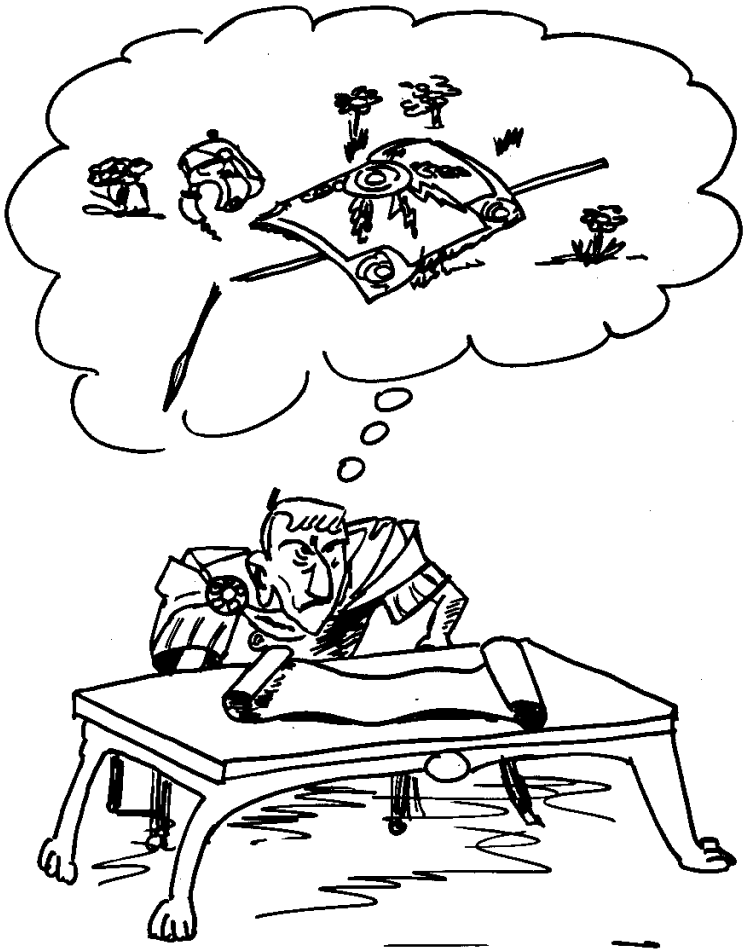
3. Extension of the alphabet
   - Classic systems
   - The Hill cipher

Outline

Outline

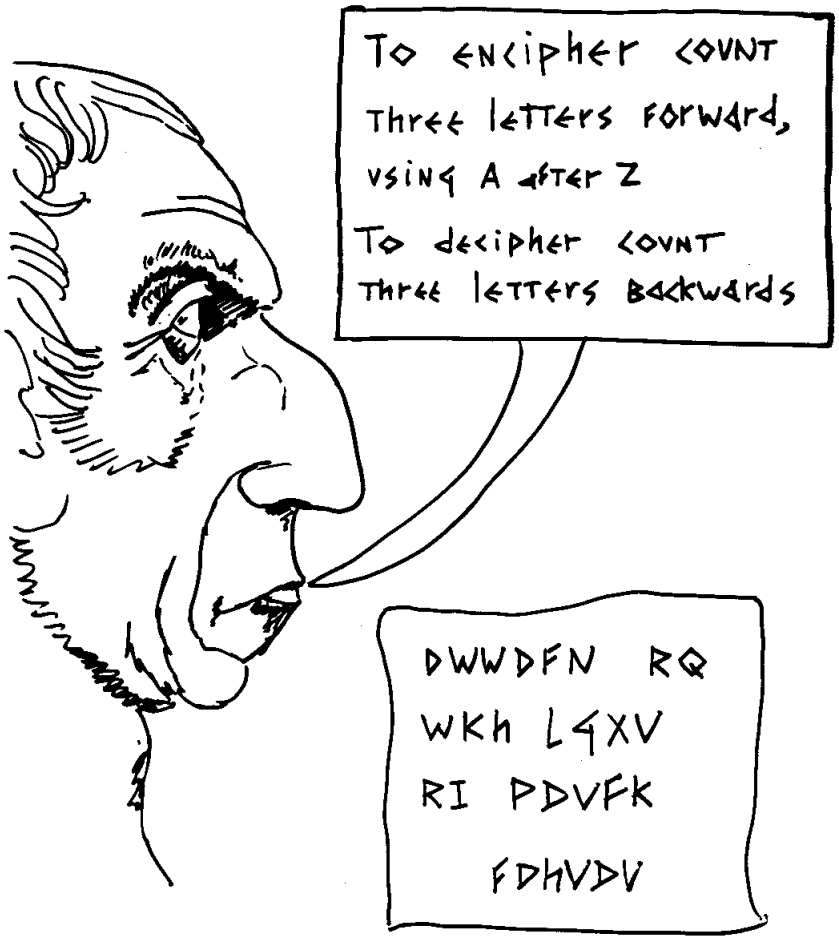
Caesar wants to hide his plans





Source: Slides Hans van der Meer

Caesar's cryptosystem



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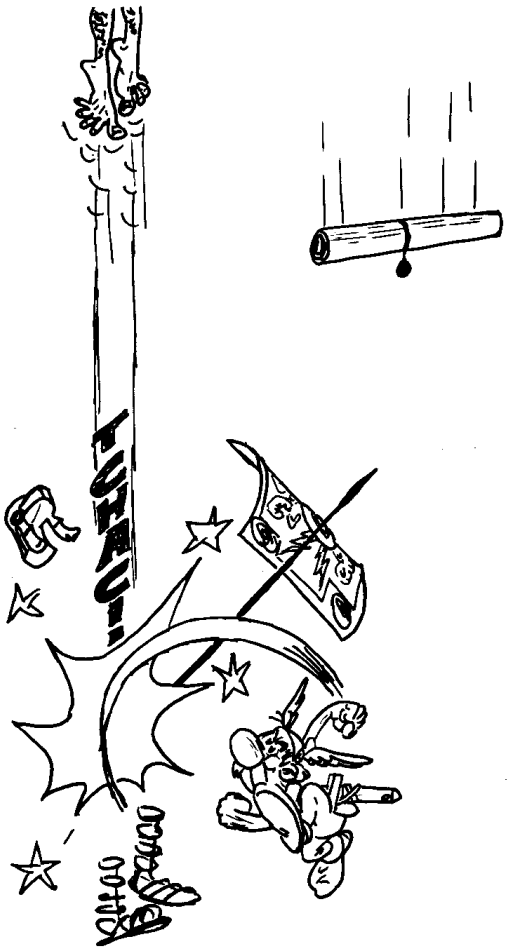
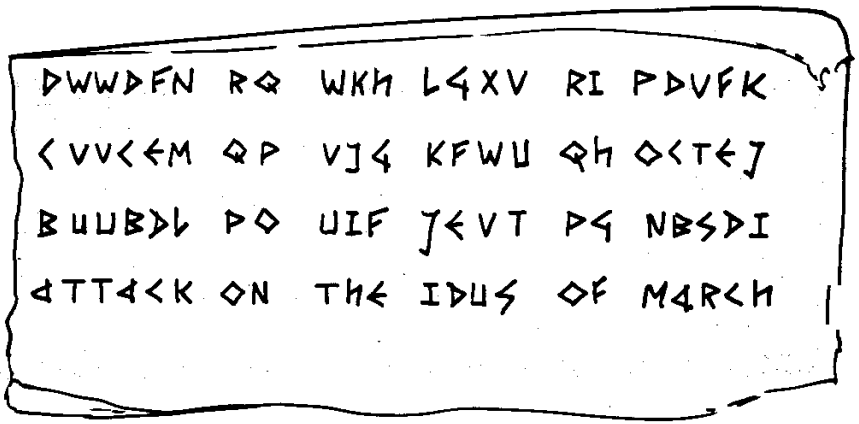
Interception and cryptanalysis





Who notices the peculiarities here?

Source: Slides Hans van der Meer

Caesar encryption

- Caesar encryption is a forward[1] rotation of the alphabet by 3 places

```
abcdefghijklmnopqrstuvwxyz
DEFGHIJKLMNOPQRSTUVWXYZABC
```

Figure 1: Rotation by 3 positions

- An example encryption

```
an example encryption
DQ HADPSOH HQFUBSWLRQ
```

Figure 2: Encryption of "an example encryption"

[1] Although historically, Suetonius mentions backward

Caesar decryption

- Caesar decryption works by turning around the encryption process

```
DEFGHIJKLMNOPQRSTUVWXYZABC

abcdefghijklmnopqrstuvwxyz
```

Figure 3: Encryption turned around (backward rotation by 3 places)

```
ABCDEFGHIJKLMNOPQRSTUVWXYZ

xyzabcdefghijklmnopqrstuvw
```

Figure 4: The same decryption reordered

Outline

Encoding (numbering) the alphabet

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
modern	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
legacy	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

- Modern mathematics starts counting at 0
- The legacy variant, starting at 1, is equivalent to ordering the alphabet as

  zabcdefghijklmnopqrstuvwxy

- This is because, when rotating the alphabet, we consider $26 = 0$

Outline

Clock arithmetic

$24 = 0$ (or maybe $12 = 0$)

- $\mathbb{Z}_{24} = \{0, 1, 2, \ldots, 23\}$
- $23 + 1 \equiv 24 \equiv 0 \pmod{24}$

Definition ($n \in \mathbb{N}, n > 1, a, b \in \mathbb{Z}$)

$a \equiv b \pmod{n} \iff n \mid (a - b) \iff \exists k \in \mathbb{Z}(k \cdot n = (a - b))$

Theorem

"$. \equiv . \pmod{n}$" is an **equivalence** *relation on $\mathbb{Z}$ which is also a* **congruence**.

$\mathbb{Z}_n$ is the set of integers modulo n.

Corollary

Addition and multiplication can be performed $\pmod{n}$ as usual.

Clock arithmetic

Examples

$$22 + 5 \equiv 3 \pmod{24}$$

$$22 \cdot 5 \equiv 110 \equiv 14 \pmod{24}$$

$$-2 \cdot 5 \equiv -10 \equiv 14 \pmod{24}$$

$$2 \cdot 12 \equiv 24 \equiv 0 \pmod{24}$$

$$2 \not\equiv 0 \pmod{24}$$

$$12 \not\equiv 0 \pmod{24}$$

$\mathbb{Z}_{24}$ has "divisors of zero" or "zero divisors", which is considered an unwanted property in general.

Clock arithmetic

Convention

$(\mathrm{mod}\ n)$ as a function

The function application $a \pmod n$ means the unique b such that $0 \leq b < n$ and $a \equiv b \pmod n$, as a relation.

- The use of $(\mathrm{mod}\ n)$ both as a binary relation as well as a function can be confusing:

$$(a \pmod n \equiv a) \pmod n$$

$$a \pmod n = (a \pmod n)$$

Who's afraid of zero?

or the AM/PM mess

- Splitting up 24 hours as $2 \cdot 12$ hours the sensible way
  - 0:00 AM (midnight), 1:00 AM, ..., 11:59 AM
  - 0:00 PM (midday, noon), 1:00 PM, ..., 11:59 PM
- Splitting up 24 hours as $2 \cdot 12$ hours the confusing way
  - 12:00 AM (midnight), 12:59 AM, 1:00 AM, ..., 11:59 AM
  - 12:00 PM (midday, noon), 12:59 PM, 1:00 PM, ..., 11:59 PM
  - $12 \equiv 0 \pmod{12}$, but $12 \not\equiv 0 \pmod{24}$,

    so using 12 hours in this context is confusing
    - It seems that in Japan 00:00 AM (12:00 PM) is midnight

      and 12:00 AM is noon

Outline

Caesar mathematically

Caesar encryption and decryption

$$\mathcal{E}(p) = (p + 3) \pmod{26} \quad (1)$$

$$\mathcal{D}(c) = (c - 3) \pmod{26} \quad (2)$$

- This works exactly the same with modern and legacy encoding
- Encryption and decryption is **keyless**
- Algorithm must be kept secret

Caesar variants with a key

Let k be a key, where $0 \leq k < 26$.

Caesar encryption and decryption with key k

$$\mathcal{E}_k(p) = (p + k) \pmod{26} \tag{3}$$

$$\mathcal{D}_k(c) = (c - k) \pmod{26} \tag{4}$$

- Even if the algorithm is known the key protects the encryption
- Since the key space is very small a brute force search is doable
- We call this is **shift cipher** or **additive cipher**

Outline

Caesar brute force decrypting “VLONY ZILWY”

Caesar brute force decrypting “VLONY ZILWY”

vlony zilwy

uknmx yhkvx

tjmlw xgjuw

silkv wfitv

rhkju vehsu

qgjit udgrt

pfihs tcfqs

oehgr sbepr

ndgfq radoq

mcfep qzcnp

lbedo pybmo

kadcn oxaln

jzcbm nwzkm

iybal mvyjl

hxazk luxik

gwzyj ktwhj

fvyxi jsvgi

euxwh irufh

dtwvg hqteg

csvuf gpsdf

brute force

aqtsd enqbd

zpsrc dmpac

yorqb clozb

xnqpa bknya

wmpoz ajmxz

Caesar brute force decrypting "VLONY ZILWY"

vlony zilwy	mcfep qzcnp	dtwvg hqteg
uknmx yhkvx	lbedo pybmo	csvuf gpsdf
tjmlw xgjuw	kadcn oxaln	**brute force**
silkv wfitv	jzcbm nwzkm	aqtsd enqbd
rhkju vehsu	iybal mvyjl	zpsrc dmpac
qgjit udgrt	hxazk luxik	yorqb clozb
pfihs tcfqs	gwzyj ktwhj	xnqpa bknya
oehgr sbepr	fvyxi jsvgi	wmpoz ajmxz
ndgfq radoq	euxwh irufh	

Outline

Outline

Monoalphabetic substitution

Definition

A monoalphabetic substitution is the systematic replacement of letters by other letters in a one-to-one way.

Example monoalphabetic encryption and decryption

abcdefghijklmnopqrstuvwxyz	ABCDEFGHIJKLMNOPQRSTUVWXYZ
DJEHKVNIOLARUQXPYWGTCSMFZB	kzuacxsdhbejwgipnlvtmfroqy

This example was generated using a Nomcom procedure with pool size 26 on input 1, 2, ..., 16^2

[2] see RFC 3797

Intermezzo: a real example (Spanish)

```
ADHRF SID QINVJX IH XDNAJIXJHAD
VFH YINEVJ YDZEVJHJ PFO J TTDPJX
J YE PDVEHJ JTTE DNAJ HFVWD DTTJ
DN YIO QFHEAJ O NEYLJAEVJ DNLDXF
WJVDXIHJ EYLXDNEFH QIDHJ
```

1. 1-letter word `a` or `y` sometimes `o`
2. 2-letter word `u.` usually `un`
3. 3-letter word `..e` usually `que`
4. 4-letter word `abbc` usually `alli` or `ella`
5. Doubled start letter mostly `l` as in

   `llegar`, `llevar`, `lleno`, `lluvia`

Generating a monoalphabetic substitution from a keyword

abcdefghijklmnopqrstuvwxyz
KEYWORDABCFGHIJLMNPQSTUVXZ

Figure 5: Using "KEYWORD" as the keyword

abcdefghijklmnopqrstuvwxyz
REPATDLSBCFGHIJKMNOQUVWXYZ

Figure 6: Using "REPEATED LETTERS" as the keyword/keyphrase

Generating a monoalphabetic substitution using decimation

abcdefghijklmnopqrstuvwxyz
EJOTYDINSXCHMRWBGLQVAFKPUZ

Figure 7: Encoding using a **multiplicative cipher** (legacy)

abcdefghijklmnopqrstuvwxyz
AFKPUZEJOTYDINSXCHMRWBGLQV

Figure 8: Encoding using a **multiplicative cipher** (modern)

- A multiplicative cipher is also called a **decimation**

Decoding of these multiplicative ciphers

ABCDEFGHIJKLMNOPQRSTUVWXYZ
upkfavqlgbwrmhcxsnidytojez

Figure 9: Decoding of the **multiplicative cipher** (legacy)

ABCDEFGHIJKLMNOPQRSTUVWXYZ
avqlgbwrmhcxsnidytojezupkf

Figure 10: Decoding of the **multiplicative cipher** (modern)

- The encoding factor was 5. What is the decoding factor?

Mathematical description of decimation

Multiplicative encryption and decryption

$$\mathcal{E}_e(p) = ep \pmod{26} \tag{5}$$

$$\mathcal{D}_d(c) = dc \pmod{26} \tag{6}$$

- There is now a difference between modern and legacy encoding
- Modern encoding works best for programming
- d is the **multiplicative inverse**[3] of e

[3] Does this always exist?

Outline

Greatest common divisor

An example of Euclid's algorithm

We want to find the gcd (greatest common divisor) of 49 and 35:

Euclid's reduction

$$49 = 1 \cdot 35 + 14 \implies \gcd(49, 35) = \gcd(35, 14)$$

$$35 = 2 \cdot 14 + 7 \implies \gcd(35, 14) = \gcd(14, 7)$$

$$14 = 2 \cdot 7 + 0 \implies \gcd(14, 7) = \gcd(7, 0) = 7$$

Euclid's reversal

$$7 = 35 - 2 \cdot 14 \quad \wedge \quad 14 = 49 - 1 \cdot 35$$

$$\begin{aligned} 7 &= 35 - 2 \cdot (49 - 1 \cdot 35) \\ &= -2 \cdot 49 + 3 \cdot 35 \end{aligned}$$

Greatest common divisor

Euclid's algorithm

Theorem

For all $a, b \in \mathbb{Z}$ we can (effectively) find $p, q \in \mathbb{Z}$ such that

$$gcd(a, b) = p \cdot a + q \cdot b$$

Finding p and q can be done using Euclid's algorithm and reversal.

Definition

a and b are called **relatively prime** iff $\gcd(a, b) = 1$.

Theorem

If a and b are relatively prime (the extended) Euclid's algorithm calculates p and q such that

$$p \cdot a + q \cdot b = 1$$

Application to decimation

In our example we had $e = 5$ and we want to find its inverse d modulo 26.

Calculation of inverse of 5 modulo 26

$$26 = 5 \cdot 5 + 1 \implies 1 = 26 + (-5) \cdot 5$$

So the inverse of 5 modulo 26 is -5 (or 21).

- A decimation's inverse is another decimation,

  just with a different multiplication factor.
- What happens if e and 26 are not relatively prime?
- This explains why the decoding described earlier is indeed

  just a decimation with factor 21

Outline

Combining multiple ciphers

- Combining two shift ciphers with key k_1 and k_2
  - Result is shift cipher with key $k_1 + k_2$
- Combining two decimations with key e_1 and e_2
  - Result is decimation with key $e_1 e_2$
- Combining a decimation with key e and a shift with key k
  - First decimate, then shift gives the **affine cipher**

    defined by $\mathcal{E}_{e,k}(p) = ep + k \pmod{26}$
  - First shift, then decimate gives the cipher

    defined by $\mathcal{E}_{e,k}(p) = e(p + k) \pmod{26}$

    or $\mathcal{E}_{e,k}(p) = ep + ek \pmod{26}$, just another affine cipher

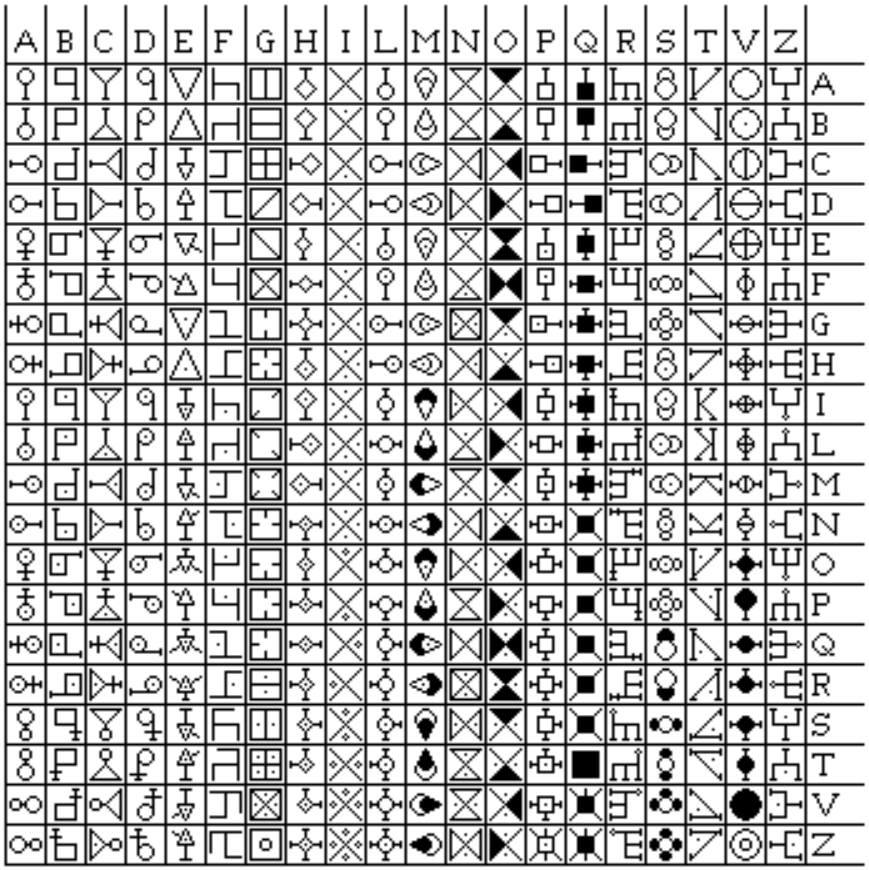
Outline

Extending the "alphabet"

- Until now substitutions are **monographic**
  - One letter of the alphabet is replaced with another letter
- What happens if we "extend the alphabet" (make it **polygraphic**)?
  - For instance replace a combination of two letters of the alphabet by another combination of two letters (so using **digraphs**)
  - Effectively this extends our alphabet from 26 to $26 \cdot 26 = 676$ "letters" (or symbols, atoms, literals, ...)
  - The number of possible (monoalphabetic) substitions increases from $26! = 403291461126605635584000000$ to $676! \approx 1.8837 \cdot 10^{1621}$

Outline

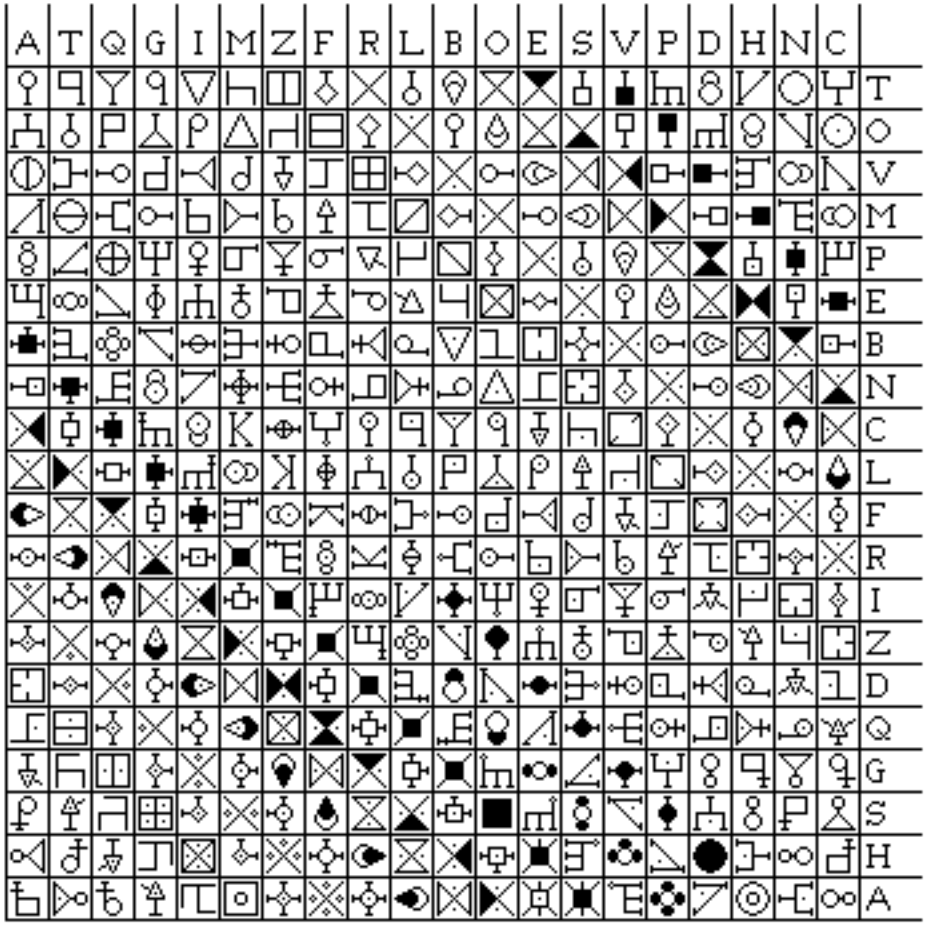
Giovanni Batista della Porta's digraph encoding



Source: `http://www.quadibloc.com/crypto/pp010302.htm`

(Can you spot anomalies?)

Giovanni Batista della Porta's digraph encoding



Source: Slides Hans van der Meer

An example digraph substitution

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	LZ	SW	BH	YJ	YR	WP	BC	FB	FW	XH	DY	MV	KC	UL	CJ	FJ	XW	BR	AD	JP	BJ	PM	JW	IU	OU	DE
B	AJ	GE	KT	AP	TN	VO	GY	CT	JS	OB	YM	MH	WJ	PF	PA	TA	IF	NR	GG	PV	LH	NX	KX	ST	UT	RP
C	LW	KW	DO	QF	JN	LX	DI	TL	DR	RM	SS	HF	RB	QU	UJ	KR	MY	GO	WF	TG	RS	YQ	SC	FI	CR	HK
D	UV	FP	PS	XZ	EV	GR	SV	KF	ZX	WL	RU	WO	YZ	JJ	NJ	VJ	IT	QT	XG	AS	KE	WE	ND	HS	YC	UH
E	AO	CZ	CI	SI	BV	OM	ZO	LE	GD	LB	OI	UK	RC	DK	PZ	YX	KJ	ZT	EM	BS	IZ	XL	RF	WA	YW	EL
F	OC	RI	SP	FY	VH	QE	SE	FC	IK	NZ	RG	LN	TX	NM	SD	JB	UQ	XY	ZG	ML	AV	JC	QM	PQ	AB	ZF
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H	BT	TK	PR	KY	EC	AN	HZ	SO	YV	MF	ES	YP	FU	AK	NI	SJ	YT	LY	TF	KV	NV	XV	DJ	WX	OO	QB
I	WR	GK	IE	QH	EZ	OY	MU	MT	LA	BP	HA	NM	TJ	QJ	AL	EE	SU	GA	HI	MG	YO	GW	KS	AY	JE	NO
J	VG	ZY	UE	FM	EH	FR	ZW	CA	DN	WD	KD	AU	GP	YS	XM	MR	NC	BQ	HC	NS	NN	ZJ	GJ	VB	RA	TH
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M	HU	EW	UC	IJ	UO	SQ	OR	EP	ZE	MX	KL	IQ	TS	QZ	BM	TI	JV	VD	XS	OH	IX	TV	TB	QN	UW	KN
N	LM	CB	SK	EY	PO	FQ	LG	MS	RK	VS	RW	CL	II	RO	ZR	NP	HX	RN	BF	IV	DX	XI	UG	BX	JM	AQ
O	TQ	XN	SH	ZS	WK	OX	WU	HH	MQ	PT	GL	QA	EX	PX	ZB	HJ	VW	SB	PL	DB	NA	CM	UX	IA	JK	LU
P	XD	GM	TC	FG	EJ	FN	WT	NF	OG	QY	DZ	NB	NU	IN	ZV	HM	CS	JU	WV	QG	FH	RQ	TE	DA	GH	AF
Q	YG	DV	EF	HV	TU	HR	LJ	CQ	FK	VC	GF	FZ	ER	XK	NW	XU	VA	ED	MN	UI	RL	GX	WH	WS	TM	OW
R	OS	XR	ID	SG	CY	TY	KG	ZN	YL	KZ	OJ	GU	VF	VR	BD	JO	GV	ZU	FF	WG	XF	GZ	KP	KU	QD	JT
S	RY	GQ	ZZ	HP	CC	HQ	UF	AD	PK	DW	XQ	DU	RH	DC	GN	QR	DM	MK	SF	RZ	MC	FT	BZ	LQ	IO	LO
T	YF	BA	UU	YN	TR	LD	WB	NQ	TW	VN	RD	FA	YU	OP	OQ	LR	FL	JI	JZ	HO	QQ	QC	GI	QW	KH	MA
U	JQ	XO	CH	EA	SY	XJ	IG	PD	ZL	LF	LP	KO	JY	ZP	UD	KA	TD	NG	ZQ	CF	AI	XT	HD	XB	UB	CW
V	XC	EI	BU	VV	AX	DF	MZ	VU	VM	RV	UP	PN	WC	FE	DT	IL	ZM	CU	EK	WZ	OF	LS	BL	IS	XA	BB
W	LI	FO	KM	JR	CV	QP	EG	WN	UA	NT	AG	UN	KK	US	WY	MP	SL	MB	BK	KB	AR	YH	DD	OE	DG	VI
X	AE	FD	ZK	SA	QX	SM	HE	CE	ZA	QV	IY	CN	PY	HN	JG	XP	AZ	UZ	BN	BW	PI	MO	AW	QL	DP	HG
Y	RX	NY	TO	MJ	SR	PE	BO	TT	BY	OV	WM	VZ	GT	CO	JL	GB	SN	NK	OL	PU	EU	RE	PP	RT	AM	CG
Z	ON	ME	IP	PB	WI	EB	LV	PW	EN	VL	NL	AA	QS	WW	RR	SZ	DQ	UM	CP	TP	IW	YK	CK	OZ	FV	IR

Source: Slides Hans van der Meer

(Can you spot anomalies?)

Playfair square with keyword

S	T	R	A	N
D	B	L	C	E
F	G	H	I	K
M	O	P	Q	U
V	W	X	Y	Z

Figure 11: Playfair square (keyword STRANDBAL) (Charles Wheatstone, 1854)

Source: Slides Hans van der Meer

Playfair (row based) substitutions

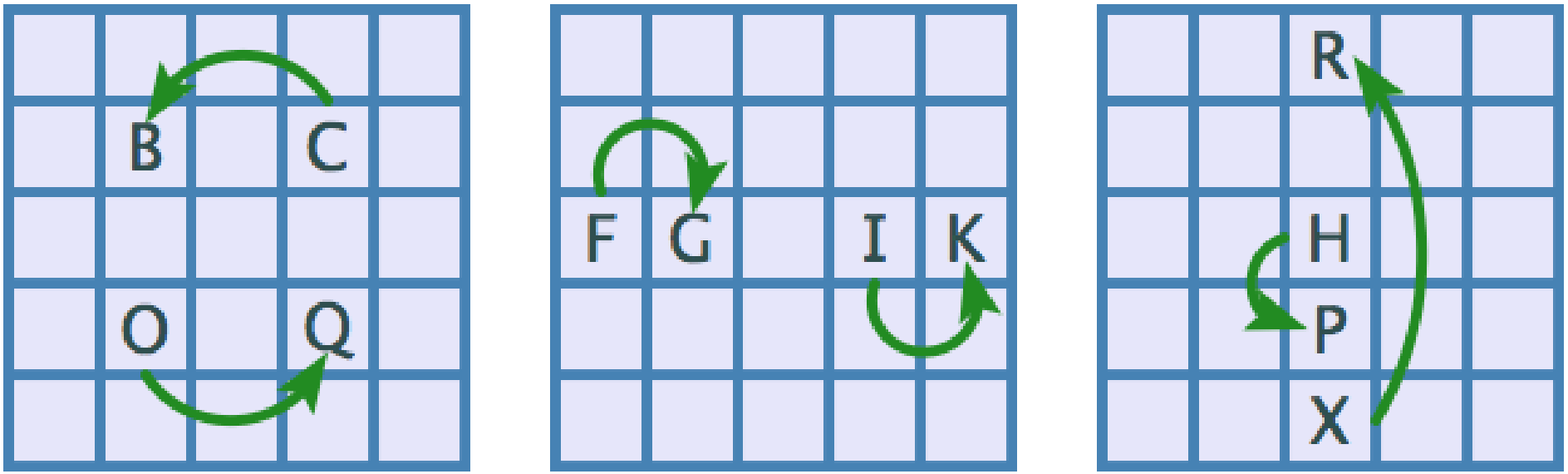

Figure 12: Playfair encryption (OC→QB; FI→GK; HX→PR)

Source: Slides Hans van der Meer

Outline

The (affine) Hill cipher

- Based on linear algebra
- Considers polygraphs as vectors
- An affine cipher built from
  - An (invertible) matrix
  - A translation vector
  - All modulo the size of the base alphabet

$$\begin{pmatrix} 3 & 5 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} \pmod{26}$$

Decoding the Hill cipher uses inverse matrix

- Encoding

$$\mathcal{E}(p_1, p_2) = \begin{pmatrix} 3 & 5 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \pmod{26}$$

- Decoding

$$\mathcal{D}(c_1, c_2) = \begin{pmatrix} -1 & 5 \\ 6 & -3 \end{pmatrix} \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \pmod{26}$$